SOLUTION OF HEAT-CONDUCTION PROBLEM WITH VARIABLE HEAT-EXCHANGE COEFFICIENT

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Exact solution is obtained in the form of an infinite series of the heat-conduction equation with boundary condition of the third kind and time-variable heat-exchange coefficient.

Considerable attention has been focused lately on solving problems of nonstationary heat conduction with variable heat-exchange coefficient [1-10], their importance in practical application being the main reason.

The mathematical formulation of the problem in the case of a planar body is as follows:

$$\frac{\partial\Theta(X,\ \mathrm{Fo})}{\partial\,\mathrm{Fo}}=\frac{\partial^2\Theta(X,\ \mathrm{Fo})}{\partial X^2}, \tag{1}$$

$$\frac{\partial\Theta(1,\ \mathrm{Fo})}{\partial X}=\mathrm{Bi}(\mathrm{Fo})\left[\Theta_c(\mathrm{Fo})-\Theta(1,\ \mathrm{Fo})\right], \tag{2}$$

$$\frac{\partial\Theta(0,\ \mathrm{Fo})}{\partial X}=0, \tag{3}$$

$$\Theta(X,\ 0)=0. \tag{4}$$

It is not possible to obtain an exact solution of Eq. (1) subject to conditions (2)-(4) in which the function Bi(Fo) is arbitrary. Various approximate methods can be found in the previously cited articles.

We will consider here the case of great interest in practice in which the function Bi(Fo) can be represented in the form

$$\mathrm{Bi}(\mathrm{Fo})=C_0-f_0(\mathrm{Fo}), \tag{5}$$

where C_0 = const and the Laplace transformation of the function f_0(Fo) is a rational function which vanishes at infinity, that is, the function f_0(Fo) can be represented as a rational combination of sines, cosines, polynomials and exponentials; then it is possible to obtain an exact solution of the problem (1)-(4) in the form of an infinite series. To this end, the method is used of "bifrequency transfer function" which was applied in the analysis of control systems described by differential equations with time-variable coefficients [11]. A brief description of this method used for solving the problem (1)-(4) is given in the Appendix.

An ordinary differential equation for the function Θ(1, Fo) will now be obtained from (1)-(4).

Let us denote the right-hand side of the condition (2) by q(Fo)

$$q(\mathrm{Fo})=\mathrm{Bi}(\mathrm{Fo})\left[\Theta_c(\mathrm{Fo})-\Theta(1,\ \mathrm{Fo})\right], \tag{6}$$

or, by inserting (6) into (2)

$$\frac{\partial\Theta(1,\ \mathrm{Fo})}{\partial X}=q(\mathrm{Fo}). \tag{7}$$

By solving Eq. (1) subject to the boundary conditions of the second kind (7) and (3) one obtains [12]

$$\frac{\overline{\Theta}(1,\ s)}{\overline{q}(s)}=\frac{1}{\sqrt{s}\,\mathrm{th}\sqrt{s}}, \tag{8}$$

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where

$$\bar{\Theta}(1, s) = \underset{\mathrm{Fo}\to s}{L}\, \Theta(1, \mathrm{Fo}), \quad \bar{q}(s) = \underset{\mathrm{Fo}\to s}{L}\, q(\mathrm{Fo}).$$

By expanding tanh √s in (8) into a series and multiplying it by √s one obtains

$$\frac{\bar{q}(s)}{\bar{\Theta}(1, s)} = a_1 s + a_2 s^2 + \ldots + a_n s^n + \ldots \tag{9}$$

In the series (9) the coefficients a_n, n = 1, 2, . . ., are constant, their actual numerical values being of no interest to us.

By taking the inverse Laplace transform of (9) and using (6) an ordinary differential equation for Θ(1, Fo)

$$a_0(\mathrm{Fo})\,\Theta(1, \mathrm{Fo}) + \sum_{n=1}^{\infty} a_n \frac{d^n}{d\,\mathrm{Fo}^n}\Theta(1, \mathrm{Fo}) = b_0(\mathrm{Fo})\,\Theta_c(\mathrm{Fo}) \tag{10}$$

is obtained where

$$a_0(\mathrm{Fo}) \equiv b_0(\mathrm{Fo}) = \mathrm{Bi}(\mathrm{Fo}). \tag{11}$$

One obtains the solution of the original problem (1)-(4) from the solution of Eq. (10) as the inverse Laplace transform of the following expression [12]:

$$\bar{\Theta}(X, s) = \frac{\mathrm{ch}\sqrt{s}X}{\mathrm{ch}\sqrt{s}}\,\bar{\Theta}(1, s). \tag{12}$$

It can be shown that the left-hand side of (10) is uniformly convergent and this enables one to use the method described in [11] to solve the equation (see Appendix).

Using the formula (A.2) the function $\alpha_k(\tau)$ is determined

$$\alpha_0(\tau) = a_0(\tau), \quad \alpha_k(\tau) = a_k, \quad k = 1, 2, \ldots \tag{13}$$

By employing (5) and (11) and denoting the constant coefficients of Eq. (10) by C_k one obtains

$$\alpha_0(\tau) = C_0 - f_0(\tau), \quad \alpha_k = C_k, \quad k = 1, 2, \ldots \tag{14}$$

It follows from (11) that the function

$$\beta_0(\tau) \equiv \alpha_0(\tau). \tag{15}$$

Using the formula (A.6) together with (13) and (14) the function Ψ(s) is formed, namely

$$\Psi(s) = C_0 + \sum_{k=1}^{\infty} a_k s^k, \tag{16}$$

or, in accordance with the expansion (9),

$$\Psi(s) = C_0 + \sqrt{s}\,\mathrm{th}\sqrt{s}. \tag{17}$$

The zeroth term (A.5) of the series (A.4) is

$$W_0(s, p) = \frac{B_0(p)}{C_0 + \sqrt{s}\,\mathrm{th}\sqrt{s}}, \tag{18}$$

where

$$B_0(p) = \frac{C_0}{p} - F_0(p), \quad F_0(p) = \underset{\tau\to p}{L} f_0(\tau). \tag{19}$$

In the case under consideration the bifrequency transfer function (A.10) is given by

$$W_u(s, q) = \frac{F_0(q)}{C_0 + \sqrt{s}\,\mathrm{th}\sqrt{s}}. \tag{20}$$

Employing the expressions (18) and (20) and also the formula (A.9) one can now find any term of the series (A.4) and obtain subsequently by using the formula (A.12) the solution of Eq. (10) in the image domain.

An example will now be given. Let

$$\Theta_c = 1,\ \mathrm{Bi}(\mathrm{Fo}) \doteq \mathrm{Bi}_2 - (\mathrm{Bi}_2 - \mathrm{Bi}_1)\exp(-\lambda\,\mathrm{Fo}).$$

From (18) and (20) one obtains

$$W_0(s,\ p) = \left(\frac{\mathrm{Bi}_2}{p} - \frac{\mathrm{Bi}_2 - \mathrm{Bi}_1}{p+\lambda}\right)\frac{1}{\Psi_0(s)}, \tag{18'}$$

$$W_u(s,\ q) = \frac{\mathrm{Bi}_2 - \mathrm{Bi}_1}{q+\lambda}\ \frac{1}{\Psi_0(s)}, \tag{20'}$$

respectively. It was assumed in the above that

$$\Psi_m(s) = \mathrm{Bi}_2 + \sqrt{s+m\lambda}\ \mathrm{th}\sqrt{s+m\lambda},\quad m = 0,\ 1,\ \ldots \tag{21}$$

The function $W_u(s, q)$ of (20') has one simple pole $q_1 = -\lambda$ with respect to the argument q. Therefore, using (A.9) one has

$$W_\nu(s,\ p) = \frac{\mathrm{Bi}_2 - \mathrm{Bi}_1}{\Psi_0(s)}\ W_{\nu-1}(s+\lambda,\ p+\lambda),\ \nu = 1,\ 2,\ \ldots \tag{22}$$

By employing the formula (22) the bifrequency transfer function W(s, p) is obtained as

$$W(s,\ p) = \sum_{\nu=0}^{\infty}\left(\frac{\mathrm{Bi}_2}{p+\nu\lambda} - \frac{\mathrm{Bi}_2 - \mathrm{Bi}_1}{p+(\nu+1)\lambda}\right)\frac{(\mathrm{Bi}_2 - \mathrm{Bi}_1)^\nu}{\prod\limits_{m=0}^{\nu}\Psi_m(s)}. \tag{23}$$

Using (A.12) and bearing in mind that $\overline{\Theta}_c(s) = 1/s$, the solution of Eq. (10) is found in the image domain. Inserting it into (12) one obtains

$$\overline{\Theta}(X,\ s) = \frac{\mathrm{ch}\sqrt{s}X}{\mathrm{ch}\sqrt{s}}\sum_{\nu=0}^{\infty}\left(\frac{\mathrm{Bi}_2}{s+\nu\lambda} - \frac{\mathrm{Bi}_2 - \mathrm{Bi}_1}{s+(\nu+1)\lambda}\right)\frac{(\mathrm{Bi}_2 - \mathrm{Bi}_1)^\nu}{\prod\limits_{m=0}^{\nu}\Psi_m(s)}. \tag{24}$$

The expression (24) is a solution of the Laplace-transformed equation (1) and the boundary conditions (2)-(4) for the adopted functions $\Theta_c(\mathrm{Fo})$ and $\mathrm{Bi}(\mathrm{Fo})$. The solution of the problem in the time domain is given by

$$\Theta(X,\ \mathrm{Fo}) = 1 + \sum_{\nu=0}^{\infty}\sum_{m=0}^{\nu}\sum_{n=1}^{\infty}\frac{\cos\zeta_{m,n}X}{\cos\zeta_{m,n}}\ \frac{(\mathrm{Bi}_2 - \mathrm{Bi}_1)^\nu}{\Psi'_m(s_{m,n})\prod\limits_{\substack{k=0\\k\neq m}}^{\nu}\Psi_k(s_{m,n})}$$

$$\times\left[\frac{\mathrm{Bi}_2}{\nu\lambda - \zeta_{m,n}^2} - \frac{\mathrm{Bi}_2 - \mathrm{Bi}_1}{(\nu+1)\lambda - \zeta_{m,n}^2}\right]\exp(-\zeta_{m,n}^2\mathrm{Fo}). \tag{25}$$

In the above $s_{m,n}$ denote the roots equations $\Psi_m(s) = 0$, m = 0, 1, 2, . . . , which can be determined by using the formula

$$s_{m,n} = -(\mu_n^2 + m\lambda) = -\zeta_{m,n}^2,\ n = 1,\ 2,\ \ldots, \tag{26}$$

in which μ_n are the roots of the transcendental equation

$$\frac{1}{\mathrm{Bi}_2}\ \mu = \mathrm{ctg}\,\mu. \tag{27}$$

It was assumed in (25) that there are no multiple poles in (24).

In the approximate computations for the formula (25) one can only use, as, similarly in the constant criterion of Biot, a finite number of terms since with m and n increasing the exponentials diminish rapidly.

Appendix

In [11] the following ordinary differential equation is considered:

$$\sum_{n=0}^{N} a_n(t)\frac{d^n}{dt^n}\ y(t) = \sum_{m=0}^{M} b_m(t)\frac{d^m}{dt^m}\ x(t),\quad M < N. \tag{A.1}$$

The equation can be solved by using the so-called bifrequency transfer function.

If the functions

$$\alpha_k(\tau) = \sum_{n=k}^{N} (-1)^{n-k} C_n^k \frac{d^{n-k}}{d\tau^{n-k}} a_n(\tau) \tag{A.2}$$

which depend on the coefficients of the left-hand side of Eq. (A.1) can be represented in the form

$$\alpha_k(\tau) = C_k - f_k(\tau), \tag{A.3}$$

in which C_k = const, and the Laplace transforms of the functions $f_k(\tau)$ are rational functions which vanish at infinity, then the bifrequency transfer function W(s, p) can be determined as an absolutely and uniformly convergent series

$$W(s, p) = \sum_{\nu=0}^{\infty} W_\nu(s, p). \tag{A.4}$$

The zeroth term of the series is given by the formula:

$$W_0(s, p) = \frac{1}{\Psi(s)} \sum_{k=0}^{M} s^k B_k(p), \tag{A.5}$$

where

$$\Psi(s) = \sum_{k=0}^{N} C_k s^k, \tag{A.6}$$

$$B_k(p) = \underset{\tau \to p}{L}\, \beta_k(\tau), \tag{A.7}$$

$$\beta_k(\tau) = \sum_{m=k}^{M} (-1)^{m-k} C_m^k \frac{d^{m-k}}{d\tau^{m-k}} b_m(\tau), \tag{A.8}$$

and the subsequent terms are obtained from the recurrence relation

$$W_\nu(s, p) = \sum_{j=1}^{i} \frac{1}{(\nu_j - 1)!} \times \left\{ \frac{d^{\nu_j - 1}}{dq^{\nu_j - 1}} \left[(q - q_j)^{\nu_j} W_u(s, q) W_{\nu-1}(s - q, p - q) \right] \right\}_{q = q_j}. \tag{A.9}$$

In the above, i denotes the number of poles in the second argument q of the bifrequency transfer function

$$W_u(s, q) = \frac{1}{\Psi(s)} \sum_{k=0}^{N} s^k F_k(q), \tag{A.10}$$

where

$$F_k(q) = \underset{\tau \to q}{L}\, f_k(\tau), \tag{A.11}$$

and ν_j is the multiplicity of the j-th pole.

Having found the bifrequency transfer function W(s, p) one finds the solution of Eq. (A.1) in the image domain given by the formula

$$Y(s) = \sum_{j=1}^{i} \frac{1}{(\mu_j - 1)!} \times \left\{ \frac{d^{\mu_j - 1}}{dp^{\mu_j - 1}} \left[(p - p_j)^{\mu_j} W(s, p) X(s - p) \right] \right\}_{p = p_j}, \tag{A.12}$$

where the sum is extended over all i poles p_j of the second argument of the function W(s, p) and μ_j denotes the multiplicity of these poles.

NOTATION

Θ	is the temperature of the plate;
Θ_c	is the temperature of the medium;
x	is the space coordinate;
L	is the thickness of the plate;
a	is the coefficient of temperature conductivity;
λ	is the heat-conduction coefficient;
α	is the heat-exchange coefficient;
$X = x/L$	is the nondimensional coordinate;
$Fo = at/L^2$	is the Fourier number;
$Bi(Fo) = \alpha(Fo)L/\lambda$	is the Biot criterion;
t	is time.

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